MAT 210 Computing Scaled QR Decomposition (column operation method)

Recall: Scaled QR decomposition is $A = QR$
- Q is same shape as A with $\perp$ columns
- R is upper-triangular, square with $\#\text{cols}(R) = \#\text{cols}(A)$

We can compute the QR decomp. like LU:

$\Rightarrow$ Q is modified version of A computed using column operations

$\Rightarrow$ R records the column operations

Plan: Work column by column in A, left-to-right

For each column, use column operations to change all later columns to be $\perp$

(**Note:** This is the opposite of the other common method — "Gram-Schmidt" — where you make each column $\perp$ to the ones before it)

First, an example just computing Q:

EX: Find Q for $A = \begin{bmatrix} 1 & 4 & -3 \\ 1 & 6 & 9 \\ 2 & 4 & -9 \end{bmatrix}$

- Use column operations to change later columns to be perpendicular to column 1

$$\begin{bmatrix} 1 & 4 & -3 \\ 1 & 6 & 9 \\ 2 & 4 & -9 \end{bmatrix}$$

$$\text{new } c_2 = c_2 - \underbrace{\frac{c_2 \cdot c_1}{c_1 \cdot c_1} c_1}_{\text{projection}}$$

$$= c_2 - \tfrac{18}{6}\, c_1 \quad \text{column op!}$$

$$\text{new } c_3 = c_3 - \frac{c_3 \cdot c_1}{c_1 \cdot c_1} c_1$$

$$= c_3 - \left(\tfrac{-12}{6}\right) c_1 \quad \text{column op!}$$

$$\begin{bmatrix} 1 & 4-3\cdot 1 & -3+2\cdot 1 \\ 1 & 6-3\cdot 1 & 9+2\cdot 1 \\ 2 & 4-3\cdot 2 & -9+2\cdot 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 11 \\ 2 & -2 & -5 \end{bmatrix}$$

now these are perpendicular!

- Use column operations to change later columns to be perpendicular to column 2

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 11 \\ 2 & -2 & -5 \end{bmatrix}$$

$$\text{new } c_3 = c_3 - \frac{c_3 \cdot c_2}{c_2 \cdot c_2} c_2$$

$$= c_3 - \tfrac{42}{14}\, c_2 \quad \text{column op!}$$

$$\begin{bmatrix} 1 & 1 & -1-3\cdot 1 \\ 1 & 3 & 11-3\cdot 3 \\ 2 & -2 & -5-3\cdot(-2) \end{bmatrix} = \begin{bmatrix} 1 & 1 & -4 \\ 1 & 3 & 2 \\ 2 & -2 & 1 \end{bmatrix}$$

now all columns are perpendicular!

- What about the matrix R ?

Just like with LU, R records the multipliers

$$\begin{bmatrix} 1 & 4 & -3 \\ 1 & 6 & 9 \\ 2 & 4 & -9 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 11 \\ 2 & -2 & -5 \end{bmatrix}\begin{bmatrix} 1 & \boxed{18/6 \quad -12/6} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{c_1 \cdot c_2}{c_1 \cdot c_1} = 18/6 \qquad \frac{c_1 \cdot c_3}{c_1 \cdot c_1} = -12/6 \qquad \frac{c_2 \cdot c_3}{c_2 \cdot c_2} = 42/14$$

These go in row 1 of R

This goes in row 2

$$= \begin{bmatrix} 1 & 1 & -4 \\ 1 & 3 & 2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 42/14 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -4 \\ 1 & 3 & 2 \\ 2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Check:
- Product of QR is $A = \begin{bmatrix} 1 & 4 & -3 \\ 1 & 6 & 9 \\ 2 & 4 & -9 \end{bmatrix}$.
- Columns of Q are all orthogonal.

Column operations making columns $\perp$ to column i of A are recorded in row i of R

For example $\dfrac{c_2 \cdot c_3}{c_2 \cdot c_2}$ goes in $\begin{cases} \text{row } 2 \\ \text{column } 3 \end{cases}$

EX: Calculate scaled QR for $A = \begin{bmatrix} -2 & -3 & 4 \\ -4 & -7 & 8 \\ 4 & 9 & -7 \\ 1 & 4 & -6 \end{bmatrix}$

$$\begin{bmatrix} -2 & -3 & 4 \\ -4 & -7 & 8 \\ 4 & 9 & -7 \\ 1 & 4 & -6 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ -4 & 1 & 0 \\ 4 & 1 & 1 \\ 1 & 2 & -4 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{c_1 \cdot c_3}{c_1 \cdot c_1} = \frac{-74}{37} = -2 \qquad \frac{c_1 \cdot c_2}{c_1 \cdot c_1} = \frac{74}{37} = 2 \qquad \frac{c_2 \cdot c_3}{c_3 \cdot c_3} = -7/7 = -1$$

$$\begin{bmatrix} -2 & 1 & 1 \\ -4 & 1 & 1 \\ 4 & 1 & 2 \\ 1 & 2 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

— New operation!! —

EX: Calculate scaled QR for $A = \begin{bmatrix} 2 & 4 & 4 \\ 2 & 3 & 4 \\ 2 & 2 & 8 \\ 2 & 3 & 4 \end{bmatrix}$

$$\begin{bmatrix} 2 & 4 & 4 \\ 2 & 3 & 4 \\ 2 & 2 & 8 \\ 2 & 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 4 \\ 1 & 3 & 4 \\ 1 & 2 & 8 \\ 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Divide column by 2 $\rightsquigarrow$ Multiply row by 2

$$\frac{c_1 \cdot c_3}{c_1 \cdot c_1} = \frac{20}{4} = 5 \qquad \frac{c_1 \cdot c_2}{c_1 \cdot c_1} = \frac{12}{4} = 3$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 3 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$-4/2 = -2$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Just like with LU we can compute faster if we calculate only Q and then write R afterwards...

EX: Calculate scaled QR for $A = \begin{bmatrix} -4 & -6 & -9 \\ -3 & -9 & 7 \\ 3 & 5 & 7 \\ 2 & 5 & 1 \end{bmatrix}$

$$\begin{bmatrix} \boxed{\begin{matrix} -4 \\ -3 \\ 3 \\ 2 \end{matrix}} & \begin{matrix} -6 \\ -9 \\ 5 \\ 5 \end{matrix} & \begin{matrix} -9 \\ 7 \\ 7 \\ 1 \end{matrix} \end{bmatrix} \rightsquigarrow \begin{bmatrix} \begin{matrix} -4 \\ -3 \\ 3 \\ 2 \end{matrix} & \boxed{\begin{matrix} 2 \\ -3 \\ -1 \\ 1 \end{matrix}} & \begin{matrix} -5 \\ 10 \\ 4 \\ -1 \end{matrix} \end{bmatrix} \rightsquigarrow \begin{bmatrix} -4 & 2 & 1 \\ -3 & -3 & 1 \\ 3 & -1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$38/38 = 1$

$76/38 = 2$

Row 1 of R

$-45/15 = -3$

Row 2 of R

Check: these are all orthogonal

$$Q = \begin{bmatrix} -4 & 2 & 1 \\ -3 & -3 & 1 \\ 3 & -1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

Check: $QR \stackrel{?}{=} A$

... at least check bottom-right

$$\left(\begin{bmatrix} 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} = 2 - 3 + 2 \stackrel{\checkmark}{=} 1 \right)$$

probably okay...

Orthogonality check:

$$\begin{bmatrix} -4 \\ -3 \\ 3 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 2 \\ -3 \\ -1 \\ 1 \end{bmatrix} = -8 + 9 - 3 + 2 = 0 \quad \checkmark$$

$$\begin{bmatrix} -4 \\ -3 \\ 3 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} = -4 - 3 + 3 + 4 = 0 \quad \checkmark$$

$$\begin{bmatrix} 2 \\ -3 \\ -1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix} = 2 - 3 - 1 + 2 = 0 \quad \checkmark$$

(usually you can check these in your head...)

Converting Scaled QR decomp. to QR decomp.

Normal QR decomp. looks the same as scaled QR except that the columns of Q all have length 1 i.e. $c_i \cdot c_i = 1$

We use a basic fact which is easy to show (but we will not show here):

FACT: The matrix product AB is preserved by the following operation

- Divide column i of A by k
- Multiply row i of B by k

"→ Move $\times k$ from A's column to B's row"

EX

$$\begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 1 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ -1 & 3 & 3 \end{bmatrix}$$

Multiply row 2 by 2

Divide column 2 by 2

Move $\times 2$ from A column to B row

$$\begin{bmatrix} 1 & 2 & 3 \\ -2 & 4 & 1 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 2 \\ -1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ -2 & 2 & 1 \\ 3 & 3 & 9 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 4 & 2 & 4 \\ -1 & 3 & 3 \end{bmatrix}$$

EX

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ -1 & 2 & 4 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix}$$

Multiply row 3 by 2

Divide column 3 by 2

Move $\times 2$ from A column to B row

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ -1 & 2 & 4 \\ 2 & 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & 0 \\ 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & 2 & 2 \\ 2 & 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ -1 & 2 & 0 \\ 2 & 2 & 6 \end{bmatrix}$$

Convert to QR decomposition by

- dividing each column of Q by length
- multiplying rows of R by these lengths

Recall: length $\underline{c}_i$ is $\sqrt{\underline{c}_i \cdot \underline{c}_i}$

EX Convert scaled QR decomposition to QR decomposition

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \\ 2 & -3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$\underline{c}_1 \quad \underline{c}_2 \quad \underline{c}_3$

- $\underline{c}_1$ length $= \sqrt{1+1+1+4} = \sqrt{7}$
- $\underline{c}_2$ length $= \sqrt{4+4+4+9} = \sqrt{21}$
- $\underline{c}_3$ length $= \sqrt{1+4+9+0} = \sqrt{14}$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \\ 2 & -3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiply by $\sqrt{7}$

Multiply by $\sqrt{21}$

Multiply by $\sqrt{14}$

Divide by $\sqrt{7}$ — Divide by $\sqrt{21}$ — Divide by $\sqrt{14}$

$$\begin{bmatrix} 1/\sqrt{7} & 2/\sqrt{21} & 1/\sqrt{14} \\ 1/\sqrt{7} & 2/\sqrt{21} & 2/\sqrt{14} \\ 1/\sqrt{7} & 2/\sqrt{21} & 3/\sqrt{14} \\ 2/\sqrt{7} & -3/\sqrt{21} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{7} & 2\sqrt{7} & -3\sqrt{7} \\ 0 & \sqrt{21} & -2\sqrt{21} \\ 0 & 0 & \sqrt{14} \end{bmatrix}$$

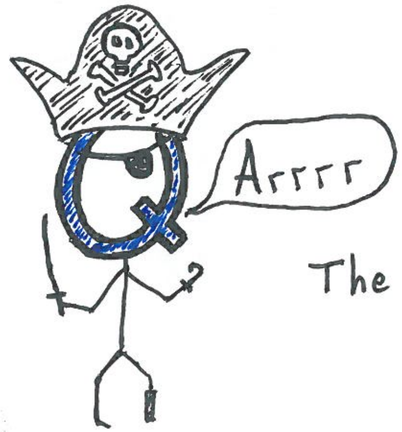
EX Convert scaled QR decomposition to QR decomposition

$$\begin{bmatrix} 1 & -4 & 0 \\ 2 & 1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

length: $\sqrt{9} = 3 \qquad \sqrt{18} = 3\sqrt{2} \qquad \sqrt{2}$

$$\begin{bmatrix} 1/3 & -4/3\sqrt{2} & 0 \\ 2/3 & 1/3\sqrt{2} & 1/\sqrt{2} \\ 2/3 & 1/3\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 3 & -9 & 12 \\ 0 & 3\sqrt{2} & 6\sqrt{2} \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

Life is too short to rationalize denominators.



The "Q Arrr" Decomposition...